A Derivation of the Reversible Moist Adiabatic Lapse Rate

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Status of this Paper

Last updated 12/18/07 to fix a couple spelling errors. Still needs formal reference style and page numbers in a few places.

1. Introduction

A rising air parcel that reaches its level of free convection (LFC) will continue to accelerate upward so long as it retains positive buoyancy, expressed mathematically as:

$$B = g\left(\frac{\theta'}{\theta_0} + 0.61 q_v - q_l\right),$$

where thermal effects, water vapor, and liquid water (which has a *negative* contribution to buoyancy) are included. It is evident from this equation that the conversion of water vapor to liquid water mass results in a decrease of buoyancy; this is the well-known "water loading" effect.

In order to calculate upward velocities, convective available potential energy[1] (CAPE), etc., an assumption must be made regarding the disposition of the water that is condensed during ascent. The two extreme assumptions are as follows (after Tsonis 2002, p. 109).

(a) All condensate products remain in the parcel. We can consider this a reversible process since the liquid will evaporate if the parcel descends. Assuming no heat transfer between the parcel and the environment, the process is also adiabatic, and thus isentropic.

(b) All condensate is immediately removed from the parcel as precipitate. The parcel always consists of "dry" air that is saturated with respect to water vapor (i.e., the relative humidity of the parcel is 100 percent). Since there is a mass transfer of the liquid away from the parcel, this is *not* a reversible process, nor is it adiabatic or isentropic. This process is defined as "pseudo-adiabatic."

Conventional treatments of the "moist adiabatic lapse rate" follow the pseudoadiabatic assumption, and lines of equivalent potential temperature drawn on Skew-T Log-P charts (AWSM, 1969) are pseudoadiabats. Many calculations of CAPE also assume upward motions are pseudoadiabatic (e.g., Barlow et al. 1998 [*others?*]). The other "extreme" of total water conservation has been stated as important in tropical environments (e.g., Xu and Emanuel 1989). The "actual amounts of condensate loading are variable, and lie somewhere between the zero loading" and significant loading assumed in these two theories (McCaul et al. 2005).

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[1] The virtual temperature correction for CAPE calculations (Doswell and Rasmussen 1994) is implicit in this derivation.

Derivation of the pseudoadiabatic moist lapse rate can be found in many texts (e.g., Tsonis 2002, Iribarne and Godson 1980, [*others?*]), and is fairly straightforward and will be avoided here for those reasons. The reversible moist adiabatic lapse rate, however, is notably more challenging because of retention of *liquid* water mass and the number of assumptions required to achieve a "final answer." While Iribarne and Godson (1980 pp. [*insert page numbers*]) offer a majority of the steps to complete the reversible MALR derivation, two factors motivate me to complete this derivation and publish it on the Internet: their book is now out of print and difficult to find; and the subject matter is of importance to any student of thermodynamics and of precipitating convection.

2. Underlying equations and assumptions

Before we step further into this derivation, it will be useful to refresh the memory with the more exact forms of thermodynamic variables, and a couple of the assumptions that will be useful to us in the next section.

a. Virtual temperature; density temperature

The virtual temperature is the temperature that dry air would have if its pressure and density were equal to those of a given sample of moist air. If the moist air contains water vapor only (no liquid), Doswell and Rasmussen (1994) note obviously that the virtual temperature will be warmer than the actual temperature because "adding water vapor to a parcel makes it less dense, what can be considered equivalent to warming the parcel." Adding liquid water, however, has the opposite effect, since the mass of the parcel is increased.

Recall that for an air parcel containing water vapor only (i.e., no liquid), its virtual temperature is expressed as:

$$T_v = T\frac{1+{}^{r_v}\!/_{\varepsilon}}{1+r_v} ,$$

where r_v is the mixing ratio of water vapor to dry air and ε is the ratio of the gas constants for dry air and water vapor, $R_d/R_v = 0.622$. If liquid water (r_l) is included, the variable is often called the "density temperature" and the equation becomes:

$$T_\rho = T\frac{1+{}^{r_v}\!/_{\varepsilon}}{1+r_v+r_l} \approx T(1+0.61r_v - r_l).$$

Note that by neglecting r_l, T_ρ simplifies to T_v; since we are dealing with a reversible process that includes the liquid phase, T_ρ is our temperature of choice. I will avoid the approximation made to get the final equation, and instead stick with the more "formal," complex fraction form. A derivation of the "vapor-only" T_v can be found in numerous texts; the rarer form that includes liquid water can be found in Emanuel (1994, pp. [*page*]).

b. The hydrostatic approximation

No surprises here. We will employ a customary form of the hydrostatic approximation (see, e.g., Holton 1992), but with the more correct T_ρ instead of T_v:

$$\frac{dp}{dz} = -g\rho = -g\frac{p}{R_d T_\rho}.$$

c. *Mixing ratio and vapor pressure*

The relationship between mixing ratio and vapor pressure will be useful in our work:

$$r = \frac{\varepsilon e}{p-e},$$

where the total pressure $p = p_d + e$. We'll also need the differential quantity *dr*, which is:

$$dr = d\left(\frac{\varepsilon e}{p-e}\right) = \varepsilon\left[d_e\left(\frac{e}{p-e}\right) + ed_p\left(\frac{1}{p-e}\right)\right] =$$
$$\frac{\varepsilon + r}{p-e}de - \frac{\varepsilon e}{(p-e)^2}dp.$$

d. *Latent heat laws*

The variability of the latent heat of vaporization as a function of temperature can be expressed

$$\frac{dL_v}{dT} = c_{pw} - c_{pv}.$$

We will also make use of a form of the Clausius-Clapeyron equation:

$$\frac{de}{dT} = \frac{L_v e}{R_v T^2}.$$

3. The derivation

a. *Preliminary steps*

We define the reversible moist adiabatic lapse rate beginning with a basic form:

$$\Gamma_{rm} \equiv -\frac{dT}{dz} = -\frac{dT}{dp}\frac{dp}{dz},$$

where we use the chain rule to write the lapse rate as a function of pressure, which can be expressed as a function of height. Using the hydrostatic approximation from Section 2, our initial equation becomes:

$$\Gamma_{rm} \equiv -\frac{dT}{dz} = g\frac{P}{R_d T_v}\frac{dT}{dp} \; .$$

b. Definition of a reversible adiabatic process

To find dT/dp, we must discuss briefly what constitutes a reversible adiabatic process. Conservation of certain atmospheric variables for dry adiabatic processes (that are isentropic) can be expressed via the First Law of Thermodynamics:

$$c_v\, dT + p\, dv = c_p\, dT - \alpha\, dp = 0 \; ,$$

which eventually yields the dry adiabatic lapse rate. For moist processes, the First Law is instead written (Wallace and Hobbs 1977):

$$c_p\, dT - \alpha\, dp = -L\, dr_s \, ,$$

or, by recognizing that the latent heat of vaporization (*L*) is *not* constant (and rearranging the equation),

$$c_p\, dT - \alpha\, dp + d(Lr_s) = 0 \; .$$

Dividing through by *T* and expanding α gives

$$\frac{c_p}{T}dT - \frac{R_d}{p_d}dp_d + d\left(\frac{Lr_s}{T}\right) = 0 \; ,$$

where we have also changed *p* to p_d, a more correct form since our initial equation above dealt only with *dry* processes (so we examine only the partial pressure of dry air here). This is the conservation equation for moist processes, with r_s representing the sum of the mixing ratios of all vapor and liquid water.

c. Operation on individual terms

To begin the derivation, expand the last term on the LHS and take derivatives with respect to *p*:

$$\frac{c_p}{T}dT-\frac{R_d}{p_d}dp_d+d\left(\frac{Lr_s}{T}\right)=$$

$$\frac{c_p}{T}\frac{dT}{dp}-\frac{R_d}{p_d}\frac{dp_d}{dp}+\frac{d}{dp}\left(\frac{Lr_s}{T}\right)=$$

$$\frac{c_p}{T}\frac{dT}{dp}-\frac{R_d}{p_d}\frac{dp_d}{dp}+\frac{r_v}{T}\frac{dL_v}{dp}+\frac{L_v}{T}\frac{dr_v}{dp}+L_vr_v\frac{d}{dp}\left(\frac{1}{T}\right)=$$

$$\frac{c_p}{T}\frac{dT}{dp}-\frac{R_d}{p_d}\frac{dp_d}{dp}+\frac{r_v}{T}\frac{dL_v}{dp}+\frac{L_v}{T}\frac{dr_v}{dp}-\frac{L_vr_v}{T^2}\frac{dT}{dp}.$$

1 2 3 4 5

For the sake of simplicity, let's now operate on each term individually.

Term 1: nothing to do here.

Term 2:

$$-\frac{R_d}{p_d}\frac{dp_d}{dp}=-\frac{R_d}{p_d}\left(\frac{dp}{dp}-\frac{de}{dp}\right)=-\frac{R_d}{p_d}\left(1-\frac{de}{dT}\frac{dT}{dp}\right)=-\frac{R_d}{p_d}-\frac{R_d}{p_d}\frac{L_ve}{R_vT^2}\frac{dT}{dp}$$

$$=-\frac{R_d}{p_d}-\frac{L_ve\varepsilon}{p_dT^2}\frac{dT}{dp}$$

Term 3:

$$\frac{r_v}{T}\frac{dL_v}{dp}=\frac{r_v}{T}\frac{dL_v}{dT}\frac{dT}{dp}=\frac{r_v}{T}\left(c_{pw}-c_{pv}\right)\frac{dT}{dp}$$

Term 4:

$$\frac{L_v}{T}\frac{dr_v}{dp}=\frac{L_v}{T}\frac{dr_v}{dT}\frac{dT}{dp}=\frac{L_v}{T}\left[\frac{\varepsilon+r}{p-e}\frac{de}{dT}-\frac{\varepsilon e}{(p-e)^2}\frac{dp}{dT}\right]\frac{dT}{dp}=$$

$$\frac{L_v}{T}\frac{\varepsilon+r}{p-e}\frac{de}{dT}\frac{dT}{dp}-\frac{L_v}{T}\frac{\varepsilon e}{(p-e)^2}=$$

$$\frac{L_v^2e}{R_vT^3}\frac{\varepsilon+r}{p-e}\frac{dT}{dp}-\frac{L_v}{T}\frac{\varepsilon e}{(p-e)^2}$$

Term 5: nothing to do here.

d. Finding dT/dp

Now that each term has been "simplified," we can recombine all terms and write an equation for dT/dp:

$$\frac{dT}{dp} = -\left[\frac{R_d}{p_d} + \frac{L_v}{T}\frac{\varepsilon e}{(p-e)^2}\right]\left[\frac{c_{pd} + c_{pw}r_t}{T} - \frac{L_v e\varepsilon}{p_d T^2} - \frac{r_v}{T}\left(c_{pw} - c_{pv}\right) + \frac{L_v^2 e}{R_v T^3}\frac{\varepsilon + r}{p-e} + \frac{L_v r_v}{T^2}\right]^{-1},$$

or, to simplify matters,

$$\frac{dT}{dp} = -\frac{\text{numerator}}{\text{denominator}}.$$

Additional algebraic work on the numerator yields:

$$\frac{R_d}{p_d} + \frac{L_v}{T}\frac{\varepsilon e}{(p-e)^2} = \frac{R_d}{p-e} + \frac{L_v}{T}\frac{r_v}{p-e} = \frac{1}{p-e}\left(R_d + \frac{L_v r_v}{T}\right).$$

For the denominator, we should first factor out *1/T*, combine all specific heat terms (into a variable c_{p*}) for simplicity, and perform some additional algebra. The denominator is then written:

$$T^{-1}\left[c_{pd} + c_{pw}r_t - c_{pw}r_v + c_{pv}r_v - \frac{L_v e\varepsilon}{p_d T} + \frac{L_v^2}{R_v T^2}\left(\frac{e\varepsilon}{p-e} + \frac{r_v e}{p-e}\right) + \frac{L_v r_v}{T}\right] =$$

$$T^{-1}\left[c_{pd} + c_{pw}r_l + c_{pv}r_v - \frac{L_v e\varepsilon}{p_d T} + \frac{L_v^2 r_v}{R_v T^2}\left(1 + \frac{r_v}{\varepsilon}\right) + \frac{L_v r_v}{T}\right] =$$

$$T^{-1}\left[c_{p*} - \frac{L_v e\varepsilon}{p_d T} + \frac{L_v^2 r_v}{R_v T^2}\left(1 + \frac{r_v}{\varepsilon}\right) + \frac{L_v r_v}{T}\right] =$$

$$T^{-1}\left[c_{p*} - \frac{L_v r_v}{T} + \frac{L_v^2 r_v}{R_v T^2}\left(1 + \frac{r_v}{\varepsilon}\right) + \frac{L_v r_v}{T}\right] =$$

$$T^{-1}\left[c_{p*} + \frac{L_v^2 r_v (r_v + \varepsilon)}{R_d T^2}\right].$$

Note that our new specific heat term $c_{p*} = c_{pd} + c_{pv}r_v + c_{pw}r_l$, incorporates terms for dry air, water vapor, and liquid water.

Now we can write

$$\frac{dT}{dp} = -\frac{\dfrac{1}{p-e}\left(R_d + \dfrac{L_v r_v}{T}\right)}{T^{-1}\left[c_{p*} + \dfrac{L_v^2 r_v (r_v + \varepsilon)}{R_d T^2}\right]}.$$

e. *Final work*

Using the above result in our lapse rate definition, we find:

$$
\begin{aligned}
\Gamma_{rm} \equiv -\frac{g}{R_d}\frac{T}{T_\rho}\left[\frac{\dfrac{p}{p-e}\left(R_d+\dfrac{L_v r_v}{T}\right)}{c_{p*}+\dfrac{L_v^2 r_v\left(r_v+\varepsilon\right)}{R_d T^2}}\right] &= \\
-\frac{g}{R_d}\left(\frac{1+r_t}{1+\dfrac{r_v}{\varepsilon}}\right)\left[\frac{\left(1+\dfrac{r_v}{\varepsilon}\right)\left(R_d+\dfrac{L_v r_v}{T}\right)}{c_{p*}+\dfrac{L_v^2 r_v\left(r_v+\varepsilon\right)}{R_d T^2}}\right] &= \\
-g\frac{\left(1+r_t\right)\left(1+\dfrac{L_v r_v}{R_d T}\right)}{c_{p*}+\dfrac{L_v^2 r_v\left(r_v+\varepsilon\right)}{R_d T^2}}&.
\end{aligned}
$$

This is the *reversible moist adiabatic lapse rate:*

$$
\boxed{\Gamma_{rm} \equiv -g\frac{\left(1+r_t\right)\left(1+\dfrac{L_v r_v}{R_d T}\right)}{c_{pd}+c_{pw}r_l+c_{pv}r_v+\dfrac{L_v^2 r_v\left(r_v+\varepsilon\right)}{R_d T^2}}.}
$$

f. *Simplifications to the R-MALR*

If the presence of liquid water is neglected, we produce the *pseudoadiabatic lapse rate*,

$$
\boxed{\Gamma_{ps} \equiv -g\frac{\left(1+r_v\right)\left(1+\dfrac{L_v r_v}{R_d T}\right)}{c_{pd}+c_{pv}r_v+\dfrac{L_v^2 r_v\left(r_v+\varepsilon\right)}{R_d T^2}}.}
$$

This is the lapse rate used to define pseudoadiabats on most thermodynamic diagrams. We can further simplify if we wish, realizing that $r_v << 1$ in the numerator, $r_v << \varepsilon$ in the denominator, and the contribution to the specific heat from the vapor (i.e., the $c_{pv}r_v$ term) is negligible compared to c_{pd}. These three approximations yield the common *moist adiabatic lapse rate*,

$$\boxed{\Gamma_m \equiv -g\frac{\left(1+\dfrac{L_v r_v}{R_d T}\right)}{c_{pd}+\dfrac{L_v^2 r_v \varepsilon}{R_d T^2}}.}$$

Note that all my lapse rate equations have preserved a minus sign on the right-hand side; you are free to define your lapse rates without it but I prefer to explicitly point out the typical decrease in temperature with height.

4. Application

The three lapse rates above are now calculated for two different atmospheric environments: a moist, tropical one (r_t = 20 g kg^{-1}) in which most (80%) of the water mass has been converted to liquid, and a more continental regime (r_t = 10 g kg^{-1}) where only 20% of the total water is liquid. Assume, in both cases, that T = 300 K.

For the tropical case, the results are:

Γ_{rm} = 6.90 C km^{-1}
Γ_{ps} = 7.09 C km^{-1}
Γ_m = 7.11 C km^{-1}

For the continental case, we get:

Γ_{rm} = 5.77 C km^{-1}
Γ_{ps} = 5.78 C km^{-1}
Γ_m = 5.81 C km^{-1}

The presence of more vapor (regardless of the presence of liquid water) will *lessen* the lapse rate (thus increasing the positive area if a parcel curve is traced on a sounding). The presence of more liquid water moves the lapse rates "back" toward dry adiabatic conditions. Noting that Γ_{ps} and Γ_m both neglect the presence of liquid water, an air parcel that has converted all its vapor into liquid form would have a lapse rate equal to the dry adiabatic lapse rate if these equations are used; this is not correct!

5. Choice of constants

So many constants are a function of laboratory experiments and simple approximations! Even constants as simple as our gas constants are subject to fluctuations based on the choice of the universal gas constant, R^*, which is the Boltzmann constant multiplied by Avogadro's number; a quick internet search yields *six different values* on *six different web sites*!

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